Planetary Motion

Differential Equations-Extra Credit

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Planetary motion is handled similarly to that of particle motion and can help us to understand our solar system better. Differential equations lets us model planetary motion and orbital movement. Newton helped prove through his study that our planets move in elliptical patterns around the sun. According to Kyriacos Papadatos, “After many years of intense study, and guided perhaps by Kepler's empirical laws of planetary motion, he discovered that the force responsible for the planetary trajectories was no other than the force, which is commonly known as gravity” (3). In this paper we will cover Newton’s law of universal gravitation, the third law of Kepler’s planetary motion, and Newton’s Inverse Square Law from Kepler’s Laws.

Newton’s law of universal gravitation states, “a particle attracts every other particle in the universe using a force that is directly proportional to the product of their masses but also inversely proportional to the square of the distance between them.” Let M be the mass of the sun and m be the mass of a planet. If we consider the sun as fixed, its mass M as concentrated at its center, the planet as a particle, and sun and planet as isolated bodies, then by Newton’s law of universal gravitation, $F = -\frac{GMm}{r^2}$, where r is the distance of the planet from the sun’s center, and G is a proportionality constant call the gravitational constant. Planetary motion is the exact same as particle motion, therefore we can say the following:

1. A planet moves in a plane
2. The orbit of a planet follows this equation

$$r = \frac{h^2}{K(1 + e\cos\theta)},\ h \neq 0,$$

where h is the angular momentum of the planet per unit mass, e is the eccentricity of its orbit and K is the product of the gravitational constant G and the mass M of the sun.

3. The planets satisfy the law of conservation of angular momentum.
4. The planets sweep out equal areas in equal intervals of time.
5. The force field in which the planets move is conservative and satisfy the law of conservation of energy.
6. The sun is at one focus of the planet's orbit.

These attributes of planetary motion help us to solve complex problems using differential equations.

We can utilize the knowledge we know from differential equations to evaluate and understand the motion of planets and more specifically Kepler's three laws of planetary motion. These are as follows: 1) all planets move about the Sun in elliptical orbits, having the Sun as one of the foci, 2) a radius vector joining any planet to the Sun sweeps out equal areas in equal lengths of time, and 3) the squares of the sidereal periods (of revolution) of the planets are directly proportional to the cubes of their mean distances from the Sun (Britannica).

Furthermore, let's use differential equations to prove the third law of Kepler's planetary motion. As we should know, planets in space orbit around the sun. In this problem, we will set the focus of the planetary orbit on a coordinate system as (0,0). The orbit of a planet is given in the following equation: $r = \frac{h^2}{K(1+e\cos\theta)}$. The width of the ellipse (planetary orbital motion) is expressed as $L = \frac{h^2}{K}$. The ellipse is given in the equation, with respect to the origin (0,0): $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Furthermore, this leads us to another solution for the width: $L = \frac{b^2}{a}$. Now, we can substitute the new value for the width (L) into the equation we have for the ellipse. By doing this exercise, we can obtain the equation: $b^2 = \frac{ah^2}{K}$. By now implementing our initial condition of A=0 and t=0, we can get the area equation: $A = \frac{1}{2}ht$. The area of the planet is expressed as “A”, while the period of a planet’s orbit is “t”. All in all, when the planet has made a complete orbit, we obtain the final equation: $T^2 = \frac{4\pi^2 a^3}{K}$. Through the use of these differential equations techniques, we have that the time period of a planetary motion is proportional to the cube of the semimajor axis of the planetary orbit.

Another good law that we could use differential equations to solve would be Newton’s Inverse Square law. In the early 16th century, Johannes Kepler analyzed astronomical date in order to develop three laws to describe how the planets rotate around the sun. Isaac Newton was concerned because even though Kepler had said that the planets orbit, Kepler had never accurately stated what causes these planets to move in the patterns that they do. Newton knew that there must be some force that caused the motion of the moon in a circular path and the planets in a elliptical path respectfully. This is when Newton decided to prove his Inverse Square Law.

Proof: By Kepler’s second law, $\frac{dA}{dt}$ is a constant. Therefore by the equation ,

(a) $$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{c}{2} \quad , r^2\theta = c,$$

Where c is a constant. Differentiation of the second equation in (a) and multiplying the result by a constant mass m, gives

(b) $$m(2rr'\theta' + r^2\theta'') = 0, \qquad m(2r'\theta' + r\theta'') = 0$$

The equation of an ellipse in polar coordinates with one focus at the origin is

(c) $$r = \frac{A}{1 + e\cos\theta},$$

Where e < 1 is its eccentricity and A is its semi focal width. Differentiation of (c) gives

(d) $$r' = \frac{Ae\sin\theta}{(1 + e\cos\theta)^2}\theta' = \frac{A^2}{(1 + e\cos\theta)^2}\frac{e\sin\theta}{A}\theta$$

Making use of (c) and (a), we can write (d) as

(e) $$r' = r^2\frac{e\sin\theta}{A}\frac{c}{r^2} = \frac{ce}{A}\sin\theta$$

Differentiating (e) and then using (a), we obtain

(f) $$r'' = \frac{c}{A}(e\cos\theta)\,\theta' = \frac{c^2}{Ar^2}(e\cos\theta)$$

Solving (c) for e cos θ, there results

(g) $$e\cos\theta = \frac{A - r}{r}$$

Substituting this value in (f), we obtain

(h) $$r'' = \frac{c^2}{Ar^2}\left(\frac{A-r}{r}\right) = \frac{c^2}{r^3} - \frac{c^2}{Ar^2}$$

The component of a force in a radial direction is, $F_r = ma_r = m(r'' - r\theta^2)$. Replace r'' by its value in (h) and θ' by its value in (a). The following result would accor

$F_r = m\left[\frac{c^2}{r^3} - \frac{c^2}{Ar^2} - \frac{c^2}{r^3}\right] = -\frac{mc^2}{Ar^2}$ we showed that the force acting on a planet is toward or away from the sun. Since m, c^2, and A are positive constants, so the last equation tells us that the force acting on a planet is directed toward the sun, and its magnitude is inversely proportional to the square of its distance from the sun.

As adequately explained, we have found many more important concepts and complex problems that using differential equations can help us to solve. We have found that we can understand Newton's law of universal gravitation, the third law of Kepler's planetary motion, and Newton's Inverse Square Law from Kepler's Laws. This information has proved to be very useful and a crucial part of understanding planetary motion and gravitational laws.

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