Particle Motion

Differential Equations

May 13, 2019

TJ Brooks, Connor Ratcliff, and Anna Van Haitsma

In this paper we will cover three types of particle motion: vertical, horizontal, and inclined. By Newton's first law of motion, "An object at rest stays at rest and an object in motion stays in motion with the same speed in the same direction unless acted upon by an unbalanced force." By his second law, the rate of change of the momentum of a body (momentum = mass x velocity) is proportional to the resultant external force F acting upon it.

Differential equations can help us solve problems involving the particle motion of a body in the vertical direction. An example of particle motion in the horizontal could be: a body is shot straight up from the surface of the moon with an initial velocity v_0. (1) Find the velocity v of the body as a function of its distance r_m from the center of the moon. (2) Find the escape velocity of the body. The radius of the moon is approximately 1080 mi; the acceleration of gravity on the surface of the moon is approximately one-sixth that of the earth. Take the outward direction from the moon as positive.

(1) ***Solution.*** We take the origin at the center of the moon, and call R the radius of the moon. We get, (a) $a = \frac{dv}{dt} = -\frac{k}{r^2} * \frac{1}{6}$. Substituting in (a), the initial conditions $r = R, a = -g$, we find $k = gR^2$. Hence (a) becomes (b) $\frac{dv}{dt} = -\frac{gR^2}{r^2} * \frac{1}{6}$. Since we wish to find v as a function of the distance r, we replace dv/dt by its equivalent value v*(dv/dr). Hence (b) becomes (c) $v\frac{dv}{dr} = -\frac{gR^2}{r^2} * \frac{1}{6}, vdv = -\frac{gR^2}{r^2} * \frac{1}{6}dr$. Integration of (c) and insertion of the initial conditions $v = v_0, r = R$, give (d) $\int_{v_0}^{v_f} v\, dv = -gR^2 * \frac{1}{6}\int_{R_0}^{r_f} \frac{1}{r^2} dr$,whose solution is (e) $v_f^2 = v_0^2 + \frac{gR^2}{3r} - \frac{gR}{3} = v_0^2 + \frac{gR}{3}\left(\frac{R}{r} - 1\right)$

(2) ***Solution.*** The body will continue to rise until v = 0. Hence by (e), (f) $0 = v_0^2 + \frac{gR}{3}\left(\frac{R}{r} - 1\right), r = \frac{gR^2}{gR - 3v_0^2}$, which is the distance the body will rise above the center of the moon if

fired with an initial velocity v_0. Subtracting R from this value will give the distance the body will rise above the moon's surface.

The body will escape the moon, i.e., it will never return to the earth, if r increases with time. This means we will never return to the moon if r increases with time. This means we want r to become infinite as the velocity v of the body approaches zero. By (e) we see that if $v_0^2 = 2gR$, then r goes to infinity as v goes to 0. Hence,

$$\begin{aligned} v_0 &= \sqrt{gR/3} = \sqrt{(32)(1080)(5280)(1/3)} \\ &= 7,799\ ft/sec = 1.48\ mi/sec,^{*}\ approx., \\ &= 5,328\ mi/hr,\ approx., \end{aligned}$$

(g)

Another type of motion that we could use some of the techniques we have learned in differential equations is particle motion in a horizontal direction. If an object moves in a horizontal direction, either on a platform or table, a frictional force develops. The frictional force is due to one of two things. First, the gravitational force of the earth pressing the object down to the platform or table, and secondly, the texture of the surface of the platform or how rough or smooth it is. The quality of the surface is characterized by the letter μ, called the coefficient of friction of the surface. However, an object may be subject to a different force as well, resisting force due to the air resistance or other medium in which it is moving. As an example of inclined motion, we can analyze would be the following:

An object on a sled is pulled by a force of 10 pounds across a frozen pond. The object and the sled weighs 64 pounds. The coefficients of static and sliding friction are negligible. However, the force of the air resistance is twice the velocity of the sled. If the sled starts from rest, find its velocity at the end of 6 seconds and the distance it has traveled in that time. What is its terminal velocity?

Solution. Here $m = \frac{64}{32} =$ 2and the force of the air resistance is given as $2v$ pounds.

Hence the differential equation of motion of the sled is

(a) $2\frac{dv}{dt} = 10 - 2v$, $\frac{dv}{dt} + v = 5$

The solution to this is

(b) $v = 5 + c_1e^{-t}$

The initial condition is t = 0, v = 0. Hence c_1 = -5, and (b) becomes

(c) $v = 5(1 - e^{-t})$

When t = 6,

(d) $v = 5(1 - e^{-5}) = 5(1 - 0.0067) = 5(0.9933) = 4.97\ ft/sec$

By (c) the terminal velocity is found to be 5 ft/sec. To find the distance x traveled in time t = 6, we integrate (c) to obtain

(e) $x = 5(t + e^{-t})$

When t = 0 and x = 0 (taking the origin at the starting point), we find, from (e), c_2= -5. Hence, (e) becomes

(f) $x = 5(t + e^{-t} - 1)$

And when t = 5,

(g) $x = 5(5 + e^{-5} - 1) = 5\ (4 + 0.0067) = 20\ ft\ approx.$

A third motion that differential equations can assist us to solve is particle motion in an inclined direction. A vector's magnitude is provided by the length of the line and the direction that the inclination of the line is going (Tenenbaum, 1985, pg. 164). We can construct a right triangle by using the vector as the hypotenuse. This triangle has sides that are parallel to the x- and y- axes of the drawn out figure of a problem. This then implies the vectors of the sides can be viewed as x- and y- components respectively. We can also find the magnitudes and the direction of the vectors by the direction of the arrowheads. An example of inclined motion we can analyze would be the following:

A toboggan with four boys on it weighs 400 lbs. It slides down a slope with a 30° incline. Find the position and velocity of the toboggan as functions of time if it starts from rest and the coefficient of sliding friction is 1/50. If the slope is 123.2 ft long, what is the velocity of the toboggan when it reaches the bottom? Neglect air resistance.

Our first step would be to find the position and velocity. The velocity of the toboggan is 15.4t found by plugging into the equation: $v = 38 + c_1 e^{-4t/13}$. The first derivative being: $-(4c_1 e^{(-4t/13)})/13$.We can now find the position of the toboggan to be 7.7t^2. This we have found by utilizing the equation $s = 38(t + \frac{13}{4} e^{-4t/13}) + c_2$. Therefore we have $v = 15.4t$ and $s = 7.7t^2$. Now, we can move onto solving for the velocity of the toboggan when it reaches the bottom of the incline. Because we found the position to be $7.7t^2$, we use this and set it equal to the equation $s = 38(t + \frac{13}{4} e^{-4t/13}) + c_2$. When we evaluate this equation, we find the velocity of the toboggan to be 61.7 ft/sec when it reaches the bottom of the incline by which it was traveling.

We can see by this example, that differential equations allow us to understand and evaluate many different areas of particle motion, such as velocity and position. We can also implement different conditions and find the solutions using these equations. This skill can be useful in solving different real-world motions and understanding the movement when a particle is in an inclined direction.

In conclusion, there are three directions of particle motion that can be analyzed and understood by using differential equations. Horizontal, vertical, and inclined motion combined allow us to evaluate almost all particle movement and motion in the physical world. We can understand things such as car movement, objects fallings, balls being tossed, planes flying, kids jumping and even toboggans!

Bibliography

Isaac Newton. (2011). *Hutchinson's Biography Database*, 1. Retrieved from http://search.ebscohost.com/login.aspx?direct=true&AuthType=shib&db=mih&AN=32229505&site=eds-live

Tenenbaum, M., & Pollard, H. (1985). *Ordinary differential equations*. New York: Dover Publications.